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LIE GROUPS OF VARIABLE CROSS-SECTION CHANNEL FLOW

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Abstract. This paper considers Lie groups of scaling transformations for the system of equations governing inviscid, compressible, quasi-one dimensional (quasi-1D) fluid flow in channels with variable cross-section. We determine the coupling between the admissibility of the group transformations and the equation of state (EOS) model used to close the system as well as the spatial and temporal dependence of the channel cross-section. The results presented extend prior group analysis performed in Ovsiannikov [32] and Boyd et al. [8] on the equations of gas dynamics for 1D flow. In the context of channel flow, the analysis in Ovsiannikov [32] and Boyd et al. [8] is only applicable for a very limited scope of channels, namely for channels possessing planar, cylindrical and spherical symmetry where the flow is truly 1D. To illustrate the extension achieved using the quasi-1D model, we consider the classical Noh problem set up in a channel with a variable area cross-section. First, a new set of canonical variables are derived using the admissible scaling transformations. Then, by re-expressing the governing equations of motion in terms of the new variables, a solvable system of ordinary differential equations (ODEs) is acquired. The resulting explicit solutions of the problem extend the previous results of Ramsey et al. [33] to solutions of the Noh problem for quasi-1D flow.

1. Introduction

Prior to the widespread implementation of computational fluid dynamics codes to aid in the understanding and prediction of the evolution of phenomena in fluid flow, reliance on approximate or exact solutions to simple models was paramount. In modern fluid dynamics, such solutions are still of great interest and importance, usually not because of their accuracy, but because they provide a benchmark to test the complex numerical methods against. For compressible fluids, smooth, one-dimensional flow is modelled using the equations of motion pertaining to mass, momentum and energy conservation of a control volume. In a few special cases, namely flows possessing planar, cylindrical and spherical symmetry, the fluid motion can be prescribed exactly using equations possessing only one spatial coordinate. In the context of fluid flow in channels, these special cases correspond to flow in constant cross-sectional area channels, wedge shaped channels and conical channels, respectively. In the latter two cases, the cross-sectional areas exhibit linear and quadratic spatial dependence, respectively.

Thanks to the work of Bernoulli [4, 5], the dynamics of additional and more complex multi-dimensional flow in channels may also be prescribed, if only approximately, using just one spatial dimension. This extension is achieved by generalising the dependence of the channel cross section on space and time while assuming the state of the flow remains uniform over the cross-section. Fluid flow successfully described in this manner is referred to as quasi-one dimensional (quasi-1D).

Although the quasi-1D approximation has limitations, use of the model is applicable to a variety of problems. For example, it is effective in cases where the streamlines of multi-dimensional flows are known apriori and the flow between the streamlines can be modelled using a streamtube with slowly varying cross-sectional area. Alternatively, as discussed in Thompson [41, p.202], it is useful for understanding acoustic wave propagation in non-uniform media, i.e., media with variations in material properties such as temperature, energy or density like those found in atmospheric or oceanic problems involving stratified flow fields. Such acoustic propagation problems can be analyzed using the methods of geometric acoustics which involves the construction of ray tubes. Along the ray tubes the quasi-1D model serves as a good approximation.

Furthermore, explicit solutions to simple problems involving the evolution of shockwaves in channels can also be derived within the quasi-1D approximation. Real world applications coupled to shockwave phenomena in channels include the design of underground mining or explosive storage tunnel networks (e.g., safety depends on effective shockwave attenuation) [17, 18, 43], the design of gas turbines and jet engines [23, 38], the design of exhaust systems connected to combustion engines and finally the ballistic science of firearms [16]. Rudinger [35] gives many additional examples in his book on schemes for the numerical simulation of non-steady channel flow. Given the endless possibilities for channel configurations, the phenomenology and dynamics of shockwaves in channels can be incredibly complex. Heilig [21] provides many visually illustrative experimental examples. For such complex problems, reliance on numerical simulation is often essential.

Verification of computational fluid dynamics solvers often involves checking the ability of the solver to accurately compute the solutions to a series of test problems for which explicit solutions have been found. The solutions to simple problems derived within the quasi-1D approximation may serve as the basis for a series of test problems. Although this may sound straight forward, obtaining explicit solutions is often extremely challenging because of the difficulty associated with solving the non-linear system of partial differential equations (PDEs) governing fluid flow. Within the context of shockwaves in channels, Chester [10, 11], Chisnell [12] and Whitham [42] discuss approximate, yet explicit, solutions for channels with discontinuous cross-sectional area changes and Guderley [19] presents iterative solution techniques relying on the method of characteristics for channels of variable cross-sectional area. A finite number of explicit solutions have also been obtained for the special planar, cylindrical and spherical geometry cases where the flow is exactly 1D. Examples include the Sedov-Taylor-von Neuman point explosion solution [6, 24, 36, 39, 40], the converging Guderley solution [20] and the Noh solution [29]. These example solutions are self-similar in nature and can be derived using dimensional arguments to identify a similarity variable which is then used to simplify the system of equations. Alternatively, these solutions can be derived within the structured framework associated with symmetry analysis (or Lie group analysis). Hutchens [22], Ramsey [34], and Axford [3] and Ramsey et al. [33], provide examples of this for the Sedov, Guderley, and Noh problems, respectively.

Special solutions known as group-invariant solutions are obtained upon the application of symmetry analysis methods [31, Chapter 3]. Contained within the broad class of group-invariant solutions are solutions of the self-similar kind derivable using dimensional analysis which correspond to scaling (stretching) group transformations. In this paper, Lie groups of scaling transformations of the system of non-linear partial differential equations (PDEs) governing quasi-1D, compressible, inviscid flow through channels with variable cross-sectional area are investigated. From the analysis, the admissibility of the transformations is coupled to the form of the cross-sectional area and the EOS closing the system. In terms of scaling transformations, this work generalizes the results of Ovsiannikov [32] and Boyd et al. [8] which previously connected the EOS to the admissible symmetries of the 1D equations of compressible gas dynamics.

The structure of the paper is as follows. In Section 2, the model system of equations governing quasi-1D fluid flow through a channel is introduced. A detailed derivation of the system is provided in Appendix A. Section 3 then outlines the symmetry analysis method applied to the equations. This includes the specification of a system of determining equations which in turn define a criterion for identifying admissible symmetry transformations. A worked example solution for the first determining equation, constructed using mass conservation, is provided in Appendix C. Following this, the analysis focuses on a subgroup of scaling transformations and restrictions on the possible cross-sectional area function and the EOS model for the attainment of scaling solutions are discussed. At the end of Section 3, the governing system is simplified through the method of symmetry reduction and in Section 4 a solution to the classical Noh problem is determined by solving the reduced system subject to a particular function for the channel cross-section.

2. Conservation equations for channel flow

The following system of PDEs provides a simplistic model governing quasi-1D, unsteady, ideal, fluid flow through a channel in the absence of body forces

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial r}(\rho A u) = 0, \tag{2.1a}$$

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial r}\left(\rho A u^2 + pA\right) = p\frac{\partial A}{\partial r}, \tag{2.1b}$$

$$\frac{\partial}{\partial t}\left(\rho A e + \frac{\rho A u^2}{2}\right) + \frac{\partial}{\partial r}\left(\rho A u e + \frac{\rho A u^3}{2} + pAu\right) = -p\frac{\partial A}{\partial t}, \tag{2.1c}$$

where "ideal" refers to the neglect of heat conduction and viscous effects, and r, t, ρ, u, p, e, and A denote the spatial position along the channel, time, density, velocity, pressure, specific internal energy, and the cross-sectional area of the channel, respectively. Each equation in System (2.1) is derived by accounting for the change in mass, momentum and total energy, respectively within a time varying control volume such as the one pictured in Figure 1. A derivation of the equations is provided in Appendix A.

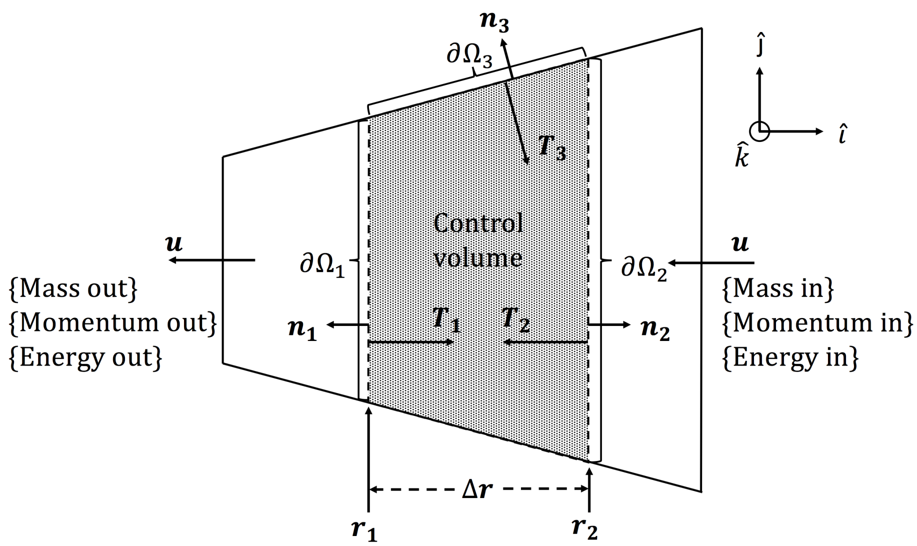

Figure 1. Flow though a control volume in a channel

As written, System (2.1) is comprised of three equations with five unknown variables and is therefore underdetermined. In order to close the system, specification of an auxiliary function describing the evolution of the cross-sectional area in space and time, $A := A(r,t)$, as well as a material EOS relating specific internal energy to density and pressure, $e := e(\rho, p)$, are required. In the subsequent discussion, the auxiliary functions are left unspecified for as long as possible. Only in the worked example section, Section 4, are particular models specified for the cross-section and EOS ensuring the general results of Section 3 are applicable to a variety of problems.

The equations in System (2.1) are presented in conservative form. The non-conservative form of the momentum balance equation

$$\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = 0, \tag{2.2}$$

commonly appearing in fluid mechanics texts [25,41] is recovered from Equation (2.1b) by expansion of the partial derivatives using the chain rule, factoring out the continuity equation wherever possible and simplifying. Similarly, it can be shown that Equation (2.1c) representing energy balance is equivalent to a statement of isentropic flow

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0, \tag{2.3}$$

where we use s to denote the specific entropy, $\mathrm{d}(*)/\mathrm{d}t$ for the total derivative

$$\frac{\mathrm{d}g(r,t)}{\mathrm{d}t} = \frac{\partial g(r,t)}{\partial t} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\partial g(r,t)}{\partial r}, \tag{2.4}$$

and $g(r,t)$ is an arbitrary function of space and time. This is not to be confused with homentropic flow for which

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \nabla s = 0. \tag{2.5}$$

The specific entropy is related to specific internal energy, pressure and density using the Gibbs equation

$$\mathrm{d}e = T\mathrm{d}s + \frac{p}{\rho^2}\mathrm{d}\rho. \tag{2.6}$$

The equivalence of Equations (2.1c) and (2.3) is to be expected because no mechanisms to facilitate entropy increase have been incorporated in the model. Details of the equivalence can be found in Appendix B.

3. Symmetry analysis

Having introduced the system of governing equations in the previous section, we proceed to apply symmetry analysis, also widely known as Lie group analysis, to study the system. Typically, the application of symmetry analysis results in the attainment of special types of solutions satisfying the system of governing equations. These solutions possess certain properties related to the notion of symmetry. More precisely, the action of the particular group of transformations used to construct the special solutions leaves the solutions unchanged. As a result, we aptly refer to these solutions as group-invariant solutions.

The general solution technique is comprised of a number of steps. First, we determine the Lie groups of point transformations admitted by the governing system of equations. A systematic approach exists for this purpose and is outlined in the subsequent discussion. Once determined, the transformations are identifiable with a differential operator, V, known as the infinitesimal group generator. Using V, a characteristic system is composed from which a new set of variables is constructed. The new variables correspond to the constants of integration appearing in the solutions of the characteristic system and therefore amount to some combination of the original variables. Next, through application of the chain rule and direct substitution, the governing system can be re-expressed solely in terms of the new co-ordinates. For a system of PDEs which admit a one-parameter group of transformations, the number of new variables required to describe the system evolution is typically one fewer than the number of original variables. If a reduction occurs in the number of independent variables, a simplification of the system is achieved. For example, for a system of first-order partial differential equations with two independent variables, such as System (2.1), symmetry analysis facilitates the reduction to a system of ordinary differential equations that we can solve either explicitly or numerically. The numerical approach required for a system of ODEs is far simpler than that needed to solve the original PDEs. Once obtained, a solution to the reduced system is then mapped into a solution of the original system.

Symmetry analysis methods were originally conceived by Sophus Lie [26] in the late 19th century. Further development of the subject and presentation of its key concepts can be found in the work of others such as Ovsiannikov [32] and Olver [31]. Both of these texts present advanced treatments of the subject. For initial exposure, Bluman and Anco [7], Stephani [37] and Logan [27] are excellent references. Additionally, a primer by Albright et. al [2] has been written to make these more in depth texts easily accessible to the reader.

3.1. **Lie groups of point transformations and the infinitesimal group generator.** The aim in this section is to introduce the type of symmetry groups focused on in this paper, define a criterion for the admissibility of the groups based on the system of equations being considered and show how this criterion relates to the infinitesimal group generator, V.

As defined in Olver [31, p.93], a symmetry group is a local group of transformations, G, acting on the space comprised of the independent and dependent variables for which the action of a group element, g, maps a solution of the system, $y = f(x)$, to a new solution, $y = \Psi(g, f(x))$, where Ψ denotes a smooth mapping from an open subset, $\mathcal{U}$, to the smooth manifold, M, $\Psi : \mathcal{U} \to M$. Identifying a symmetry group first requires interpreting the governing equations as functions with dependence on the vectors x, y, and z denoting the independent variables, the dependent variables, and the partial derivatives, respectively

$$F_n(x, y, z) = 0, \qquad n = 1 \dots N. \tag{3.1}$$

The vectors x, y and z have components

$$x_i, \quad y^j, \quad z_i^j = \frac{\partial y^j}{\partial x_i}, \qquad i = 1, \dots, I, \quad j = 1, \dots, J. \tag{3.2}$$

For the channel flow system, in this notation,

$$x_1 = r, \quad x_2 = t, \tag{3.3}$$

$$y^1 = \rho, \quad y^2 = u, \quad y^3 = p, \tag{3.4}$$

$$z_1^1 = \rho_r, \quad z_2^1 = \rho_t, \quad z_1^2 = u_r, \quad z_2^2 = u_t, \quad z_1^3 = p_r, \quad z_2^3 = p_t, \tag{3.5}$$

$N = 3$ and the subscripts denote the partial derivative with respect to the indicated argument. Next, by the definition of a symmetry group, if the point $[x, y, z]$ is a solution to the System (3.1), and if $g \in G$, the transformed point, $[\tilde{x}, \tilde{y}, \tilde{z}]$, is also a solution where

$$\tilde{x} = \Psi(g, x), \qquad \tilde{y} = \Psi(g, y), \qquad \tilde{z} = \Psi(g, z). \tag{3.6}$$

Consequently, symmetry groups can be identified with transformations that leave the functions F_n invariant, i.e.,

$$F_n(x, y, z) = 0 = F_n(\tilde{x}, \tilde{y}, \tilde{z}), \qquad n = 1, \dots, N. \tag{3.7}$$

In this paper, we consider symmetry groups known as local Lie groups of point transformations. Such transformations are of the form

$$\tilde{x}_j = x_j + \epsilon\eta_j(x, y) + \mathcal{O}(\epsilon^2), \tag{3.8}$$

$$\tilde{y}^i = y^i + \epsilon\phi^i(x, y) + \mathcal{O}(\epsilon^2), \tag{3.9}$$

$$\tilde{z}_j^i = z_j^i + \epsilon\zeta_j^i(x, y, z) + \mathcal{O}(\epsilon^2), \tag{3.10}$$

where the η_j and ϕ^i may be functions of x and y and the ζ_j^i can be functions of x, y and z. These transformations are smoothly parameterized by ϵ and we assume that locally each of the series on the right hand side (r.h.s) converges. It should be clear that the identity transformations are defined at $\epsilon = 0$,

$$\tilde{x}_j\big|_{\epsilon=0} = x_j, \qquad \tilde{y}^i\big|_{\epsilon=0} = y^i, \qquad \tilde{z}_j^i\big|_{\epsilon=0} = z_j^i. \tag{3.11}$$

At $\epsilon = 0$, the functions F_n are invariant under transformations of the kind defined by Equations (3.8) – (3.10) and Equation (3.7) is satisfied. However, we also require Equation (3.7) to hold locally around the identity as the value of ϵ is varied. To determine their local behaviour, the functions $F_n(\tilde{x}, \tilde{y}, \tilde{z})$ are expanded as Taylor series around $\epsilon = 0$,

$$F_n(\tilde{x}, \tilde{y}, \tilde{z}) = F(x, y, z) + \epsilon \left. \frac{\mathrm{d}F_n(\tilde{x}, \tilde{y}, \tilde{z})}{\mathrm{d}\epsilon} \right|_{\epsilon=0} + \mathcal{O}(\epsilon^2). \tag{3.12}$$

For Equation (3.7) to be satisfied, we require

$$\epsilon \left. \frac{\mathrm{d}F_n(\tilde{x}, \tilde{y}, \tilde{z})}{\mathrm{d}\epsilon} \right|_{\epsilon=0} + \mathcal{O}(\epsilon^2) = 0. \tag{3.13}$$

Applying the chain rule,

$$\left. \frac{\mathrm{d}}{\mathrm{d}\epsilon} F_n(\tilde{x}, \tilde{y}, \tilde{z}) \right|_{\epsilon=0} = \sum_{j=1}^{J} \eta_j(x, y) \frac{\partial F_n}{\partial x_j} + \sum_{i=1}^{I} \phi^i(x, y) \frac{\partial F_n}{\partial y^i} + \sum_{i,j=1}^{I,J} \zeta^i_j(x, y, z) \frac{\partial F_n}{\partial z^i_j}, \tag{3.14}$$

$$= (V + V^{(1)}) F_n(x, y, z) \tag{3.15}$$

$$= \mathrm{pr}^{(1)} V F_n(x, y, z), \qquad n = 1, \ldots, N, \tag{3.16}$$

where

$$V = \sum_{j=1}^{J} \eta_j(x, y) \frac{\partial}{\partial x_j} + \sum_{i=1}^{I} \phi^i(x, y) \frac{\partial}{\partial y^i}, \qquad V^{(1)} = \sum_{i,j=1}^{I,J} \zeta^i_j(x, y, z) \frac{\partial}{\partial z^i_j}, \tag{3.17}$$

and

$$\mathrm{pr}^{(1)} V = V + V^{(1)}. \tag{3.18}$$

The higher order terms appearing in Equation (3.12) can be expressed in the form

$$\frac{\epsilon^2}{2!} \mathrm{pr}^{(1)} V (\mathrm{pr}^{(1)} V F_n) + \frac{\epsilon^3}{3!} \mathrm{pr}^{(1)} V^2 (\mathrm{pr}^{(1)} V F_n) + \cdots . \tag{3.19}$$

As a result, it is sufficient to conclude that if

$$\left. \mathrm{pr}^{(1)} V F_n(x, y, z) \right|_{F_n=0} = 0, \tag{3.20}$$

then Equation (3.13) is satisfied and by extension so is Equation (3.7). Equation (3.20) defines the criterion for infinitesimal invariance and therefore transformations derived from this condition locally leave the equations of the system studied invariant.

The differential operator, V, defined in Equation (3.17), is the infinitesimal group generator and acts on the space of independent and dependent variables. Key components of this generator are the coefficients $\eta_j(x, y)$ and $\phi^i(x, y)$, subsequently referred to as the independent and dependent co-ordinate functions, respectively. Together the infinitesimal group generator and $V^{(1)}$ combine to form the first-order prolongation of the group generator denoted $\mathrm{pr}^{(1)} V$. The superscript (1) corresponds to the order of the prolongation which typically reflects the highest order partial derivatives appearing in the governing system. Only the prolongation to first order is required to analyse the system of channel flow equations.

The prolongation of the group generator appears in the criterion for infinitesimal invariance, as opposed to just V, as a consequence of treating the differential equations of the system as functions of the derivatives as well as the independent and dependent variables. The operator $V^{(1)}$ extends the action of V to the larger space composed of the independent variables, dependent variables and partial derivatives. The functions $\zeta^i_j(x, y, z)$ appearing in $V^{(1)}$ preserve the relationship between the independent and dependent variables, and the partial derivatives when the partial derivatives are included as additional variables. The $\zeta^i_j(x, y, z)$ are related to η_j and ϕ^i using the prolongation formula. For more information on the prolongation formula see [2, 7, 31]. To first order, the ζ_j^i are given by

$$\zeta_j^i(x,y,z) = \frac{\partial \eta^i}{\partial x_j} + \sum_{k=1}^{I} \frac{\partial \eta^i}{\partial y^k} z_j^k - \sum_{k=1}^{J} \frac{\partial \phi_k}{\partial x_j} z_k^i - \sum_{k,l=1}^{I,J} \frac{\partial \phi_k}{\partial y^l} z_k^i z_j^l. \tag{3.21}$$

Combined with Equation (3.21), Equations (3.20) define a set of determining equations that must be solved to determine the co-ordinate functions η_j and ϕ^i. The full symmetry group is determined from the general solution to these equations. However, for the purposes of this paper, the focus is instead restricted to a search for a subgroup of scaling transformations. These scaling transformations are discussed in the following section.

3.2. **Scaling transformations.** For the remainder of this paper, the point transformations considered are scaling transformations of the form

$$\tilde{r} = e^{a_1\epsilon} r, \qquad \tilde{t} = e^{a_2\epsilon} t, \tag{3.22}$$

$$\tilde{\rho} = e^{a_3\epsilon}\rho, \qquad \tilde{u} = e^{a_4\epsilon} u, \qquad \tilde{p} = e^{a_5\epsilon} p, \tag{3.23}$$

where the a_m are real numbers known as the group parameters. The transformations of the independent and dependent variables given by Equations (3.22) and (3.23), respectively correspond to the group generator

$$V = a_1 r\frac{\partial}{\partial r} + a_2 t\frac{\partial}{\partial t} + a_3\rho\frac{\partial}{\partial \rho} + a_4 u\frac{\partial}{\partial u} + a_5 p\frac{\partial}{\partial p}, \tag{3.24}$$

which has co-ordinate functions

$$\eta_1 = a_1 r, \qquad \eta_2 = a_2 t, \tag{3.25}$$

$$\phi^1 = a_3\rho, \qquad \phi^2 = a_4 u, \qquad \phi^3 = a_5 p. \tag{3.26}$$

Using Equation (3.21), the first order prolongation is

$$\begin{aligned}\mathrm{pr}^{(1)}V = {} & a_1 r\frac{\partial}{\partial r} + a_2 t\frac{\partial}{\partial t} + a_3\rho\frac{\partial}{\partial \rho} + a_4 u\frac{\partial}{\partial u} + a_5 p\frac{\partial}{\partial p} \\ & + (a_3 - a_1)\rho_r\frac{\partial}{\partial \rho_r} + (a_3 - a_2)\rho_t\frac{\partial}{\partial \rho_t} + (a_4 - a_1)u_r\frac{\partial}{\partial u_r} \\ & + (a_4 - a_2)u_t\frac{\partial}{\partial u_t} + (a_5 - a_1)p_r\frac{\partial}{\partial p_r} + (a_5 - a_2)p_t\frac{\partial}{\partial p_t}.\end{aligned} \tag{3.27}$$

3.3. **Scaling parameter, cross-sectional area and EOS constraints.** By solving the system of determining equations, Equations (3.20), (comprised of System (2.1) and the generator given by Equation (3.27)) it can be determined that the values of the group parameters cannot be set completely arbitrarily. We find the following relationships must be imposed

$$a_4 = a_1 - a_2, \tag{3.28}$$

$$a_5 = 2(a_1 - a_2) + a_3. \tag{3.29}$$

An example of solving the determining equation using the continuity equation is given in Appendix C. This result was verified using Symgrp, an open source computational symmetry analysis package developed for Maxima [1, 9]. The constraints in Equations (3.28) and (3.29) simply amount to requiring dimensional consistency between the variables as they are scaled. For example, scaling in the variables r and t by factors $e^{a_1\epsilon}$ and $e^{a_2\epsilon}$ respectively must be accompanied by a scaling in the velocity given by $e^{a_4\epsilon} = e^{a_1\epsilon}/e^{a_2\epsilon} = e^{(a_1-a_2)\epsilon}$.

Aside from the constraints of Equations (3.28) and (3.29), the admissibility of particular symmetry transformations is also contingent on the cross-sectional area of the duct as a function of space and time, $A(r,t)$, in addition to the choice of specific internal energy function, $e(\rho,p)$, acting as the EOS. The following differential equations stem from the determining system and must be satisfied

$$a_1(rAA_{rr} - rA_r^2 + AA_r) + a_2(tAA_{rt} - tA_rA_t) = 0, \tag{3.30}$$

$$a_1(rAA_{rt} - rA_rA_t) + a_2(tAA_{tt} - tA_t^2 + AA_t) = 0, \tag{3.31}$$

$$\begin{aligned} a_3 \left(\rho^2 p(e_\rho e_{pp} - e_{\rho p}e_p) - p^2 e_{pp} - \rho^3 e_{\rho\rho}e_p - pe_p + \rho^3 e_\rho e_{\rho p} - \rho p e_{\rho p}\right) \\ + 2a_2 \left(\rho^2 p(e_{\rho p}e_p - e_\rho e_{pp}) + p^2 e_{pp} - \rho^2 e_\rho e_p\right) \\ + 2a_1 \left(\rho^2 p(e_\rho e_{pp} - e_{\rho p}e_p) - p^2 e_{pp} + \rho^2 e_\rho e_p\right) = 0, \end{aligned} \tag{3.32}$$

where the subscripts again denote partial differentiation with respect to the indicated argument. In the following section, we consider some cross-sectional area and EOS models that are compatible with these constraints.

3.4. **Cross-sectional area and EOS models.** In the previous, Equations (3.30), (3.31) and (3.32) were found from the system of determining equations. Based on the values of the group parameters, these constraints restrict the admissible models for the cross-sectional area function and the EOS that permit scaling symmetries. In the subsequent discussion, some combinations of the area function, EOS and group parameter values that satisfy the constraints are presented. First, the case of a time independent area function is considered, $A := A(r)$. Consequently, Equation (3.31) is satisfied and Equation (3.30) reduces to

$$a_1(rAA_{rr} - rA_r^2 + AA_r) = 0. \tag{3.33}$$

If a_1 is equal to zero, this equality is satisfied for any cross-section. Alternatively, assuming $a_1 \neq 0$, the most general solution for the area function is given by

$$A = c_1 r^{c_2}, \tag{3.34}$$

where c_1 and c_2 are arbitrary constants. In the select cases where c_2 takes discrete values of $0, 1$ or 2, substituting this area function into System (2.1) recovers the equations governing motion in the exact planar, cylindrical and spherical geometry cases, respectively. For example, coupled with the ideal gas EOS, this cross-sectional area function gives equations analogous to those considered by Guderley in his study of a converging shockwave in curvilinear geometries [20,34]. Equation (3.34) informs us about the time independent, geometrical nature of the region bounding scale invariant fluid flow. It also illustrates the generalization achieved through use of the quasi-1D model which generalises the constants c_1 and c_2 to a continuous set of values.

Next, by including time dependence in the area function, Equations (3.30) and (3.31) become far more challenging to solve. A general solution is not provided in this case. Instead, only some possibilities along with the corresponding values of the constants a_1 and a_2 necessary for symmetry existence are discussed. The first possibility is given by

$$A = c_1 r^{c_2} t^{c_3}, \tag{3.35}$$

where c_1, c_2 and c_3 are constants. This particular example does not require any further constraints imposed between a_1 and a_2. However, we could consider adding an additional constant c_4 to obtain

$$A = c_1 r^{c_2} t^{c_3} + c_4. \tag{3.36}$$

In this case, assuming c_1 and c_4 are non-zero, it is also required that

$$a_1 c_2 + a_2 c_3 = 0, \tag{3.37}$$

and therefore the choice of constants c_2 and c_3 determine the relationship between a_1 and a_2 or vice versa. Another solution for the area function satisfying Equations (3.30) and (3.31), determined by inspection is

$$A = c_1 r^{c_2} + c_3 t^{c_2}. \tag{3.38}$$

This solution requires the values for a_1 and a_2 to be equal. In Section 4, this particular cross-sectional function will be applied to determine a solution to the example problem considered.

In addition to a function describing the cross-sectional area, solving example problems requires the prescription of an EOS. The most simplistic of these, satisfying Equation (3.32) without demanding constraints on the group parameters is the ideal gas EOS

$$e = \frac{p}{(\gamma - 1)\rho}, \tag{3.39}$$

where γ denotes the adiabatic parameter. Alternatively, by inspection, if $a_3 = 0$ and $a_1 = a_2$, Equation (3.32) is satisfied for any EOS. Note that this coinsides with the values of the group parameters required for the cross-section given by Equation (3.38).

3.5. **Symmetry reduction of the duct flow equations.** Now that the constraints on the infinitesimal group generator have been discussed, the characteristic system corresponding to the group generator in Equation (3.24) is solved to construct new variables. The new variables are used to re-express the channel flow equations. Subject to the constraints of Equations (3.28) and (3.29), the characteristic system is given by

$$\frac{\mathrm{d}r}{a_1 r} = \frac{\mathrm{dt}}{a_2 t} = \frac{\mathrm{d}\rho}{a_3 \rho} = \frac{\mathrm{d}u}{(a_1 - a_2)u} = \frac{\mathrm{d}p}{[2(a_1 - a_2) + a_3]\, p}. \tag{3.40}$$

Solving the equality consisting of the first two members of Equation (3.40), under the assumption $a_1 \neq 0$, we obtain

$$r^{a_2/a_1} = ct, \tag{3.41}$$

where c is the constant of integration. Rearranging and relabelling c using ξ

$$\xi = \frac{r^{a_2/a_1}}{t}. \tag{3.42}$$

The new variable ξ is therefore a constant contingent on the initial values of r and t and on the parameters a_1 and a_2. It will subsequently be referred to as the independent similarity variable. Similarly, from the equalities consisting of the first and third, first and fourth and first and fifth members of Equation (3.40) we find

$$\rho = \hat{\rho}(\xi) r^{a_3/a_1} = \hat{\rho}(\xi) r^{\delta_2}, \tag{3.43}$$

$$u = \hat{u}(\xi) r^{(a_1 - a_2)/a_1} = \hat{u}(\xi) r^{\delta_1}, \tag{3.44}$$

$$p = \hat{p}(\xi) r^{[2(a_1 - a_2) + a_3]/a_1} = \hat{p}(\xi) r^{2\delta_1 + \delta_2}, \tag{3.45}$$

where $\hat{\rho}, \hat{u}$ and $\hat{p}$ are integration constants and a re-parameterization has been made using $\delta_1 = (a_1 - a_2)/a_1$ and $\delta_2 = a_3/a_1$. The integration constants appearing in Equations (3.43) – (3.45) are considered to be arbitrary functions of the similarity variable and constitute the new set of dependent variables. The exact dependance of each on ξ must be determined.

Substituting Equations (3.43)–(3.45) into Equations (2.1) and noting that, by the chain rule, the partial derivatives of a function $\hat{f}(\xi)$ can be expressed as

$$\frac{\partial \hat{f}(\xi)}{\partial r} = \frac{\partial \xi}{\partial r}\frac{\partial \hat{f}(\xi)}{\partial \xi} = \frac{a_2}{a_1}\frac{r^{a_2/a_1}}{tr}\frac{\partial \hat{f}(\xi)}{\partial \xi} = (1 - \delta_1)\frac{\xi}{r}\frac{\partial \hat{f}(\xi)}{\partial \xi}, \tag{3.46}$$

$$\frac{\partial \hat{f}(\xi)}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial \hat{f}(\xi)}{\partial \xi} = -\frac{r^{a_2/a_1}}{t^2}\frac{\partial \hat{f}(\xi)}{\partial \xi} = -\frac{\xi}{t}\frac{\partial \hat{f}(\xi)}{\partial \xi}, \tag{3.47}$$

the system of channel flow equations becomes

$$A\left[-\xi\frac{\partial\hat{\rho}}{\partial\xi}+r^{\delta_1}\frac{t}{r}\left\{(\delta_1+\delta_2)\hat{\rho}\hat{u}+(1-\delta_1)\xi\frac{\partial(\hat{\rho}\hat{u})}{\partial\xi}\right\}\right]+A_t t\hat{\rho}+A_r t r^{\delta_1}\hat{\rho}\hat{u}=0, \quad (3.48)$$

$$-\xi\hat{\rho}\frac{\partial\hat{u}}{\partial\xi}+\frac{t}{r}r^{\delta_1}\left\{\delta_1\hat{\rho}\hat{u}^2+(2\delta_1+\delta_2)\hat{p}+(1-\delta_1)\xi\frac{\partial\hat{p}}{\partial\xi}+(1-\delta_1)\xi\hat{\rho}\hat{u}\frac{\partial\hat{u}}{\partial\xi}\right\}=0, \quad (3.49)$$

$$\begin{aligned}A\Bigg[r^{\delta_1}\frac{t}{r}\Bigg\{\hat{\rho}\hat{u}\left(e_p r^{2\delta_1+\delta_2}\left((2\delta_1+\delta_2)\hat{p}+(1-\delta_1)\xi\frac{\partial\hat{p}}{\partial\xi}\right)+e_\rho r^{\delta_2}\left(\delta_2\hat{\rho}+(1-\delta_1)\xi\frac{\partial\hat{\rho}}{\partial\xi}\right)\right)\\ +r^{2\delta_1}\left((1-\delta_1)\xi\hat{p}\frac{\partial\hat{u}}{\partial\xi}+\delta_1\hat{p}\hat{u}\right)\Bigg\}-r^{\delta_2}\hat{\rho}\left\{e_p r^{2\delta_1+\delta_2}\xi\frac{\partial\hat{p}}{\partial\xi}+e_\rho\xi\frac{\partial\hat{\rho}}{\partial\xi}\right\}\Bigg]\\ +A_t t r^{2\delta_1}\hat{p}+A_r t r^{3\delta_1}\hat{p}\hat{u}=0, \quad (3.50)\end{aligned}$$

For further details on the derivation of these equations, see Appendix D. Specification of the functions $A(r,t)$ and $e(\rho,p)$ are required to fully reduce the system and hence express the equations entirely in terms of the new variables.

4. Example - The Noh problem

First introduced by W. F. Noh in 1987 [29], the “Noh problem” is used to test and verify the accuracy of hydro-codes. This test problem has many benefits: it is straightforward to initialize, has a simple closed-form solution for an ideal gas, and provides numerous phenomena to test, including wall-heating effects and system asymmetries. In the following discussion, the problem setup is defined and the compatibility of the initial conditions of the problem with the symmetry transformations discussed in Section 3 are considered once again via the infinitesimal group generator. Additionally, a cross-sectional area function is specified and a solution is obtained which concurs with previous work in the literature.

4.1. **Problem definition.** In this test problem, a fluid of spatially uniform initial density $\rho_0 = \rho(r,t=0)$ flows at a constant initial velocity $u_0 = u(r,t=0)$ into a rigid wall located at $r=0$. The resulting discontinuity in the fluid velocity at impact sends a shockwave back upstream into the incoming fluid behind which the flow is assumed to stagnate. The problem setup at time $t=0$ is shown in Figure 2 for a conically shaped channel.

At some subsequent time $t>0$, the shock wave has travelled a finite distance into the incoming fluid. The problem is therefore separated into two regions: shocked and unshocked, denoted α and β respectively. The regions are displayed in Figure 3. The system of channel flow equations assume the flow is smooth and are therefore only applicable in the regions on either side of the discontinuous shockwave. Consequently, solutions in the regions α and β are determined independently of one another. In order to stitch the solutions together and account for the discontinuous change in fluid variables across the shockwave, we rely upon the Rankine-Hugoniot jump conditions [15, Chap. 16]

$$\frac{\rho_\beta}{\rho_\alpha}=\frac{D-u_\alpha}{D-u_\beta}, \quad (4.1)$$

$$p_\beta-p_\alpha=\rho_\alpha(u_\beta-u_\alpha)(D-u_\alpha), \quad (4.2)$$

$$e_\beta-e_\alpha=\frac{p_\beta u_\beta-p_\alpha u_\alpha}{\rho_\alpha(D-u_\alpha)}-\frac{1}{2}(u_\beta^2-u_\alpha^2). \quad (4.3)$$

In Equations (4.1) – (4.3), D denotes the shock front velocity

$$D=\frac{\mathrm{d}r_s}{\mathrm{d}t}, \quad (4.4)$$

where the position of the shockwave is denoted $r_s(t)$. At $t=0$, $r_s(t=0)=0$.

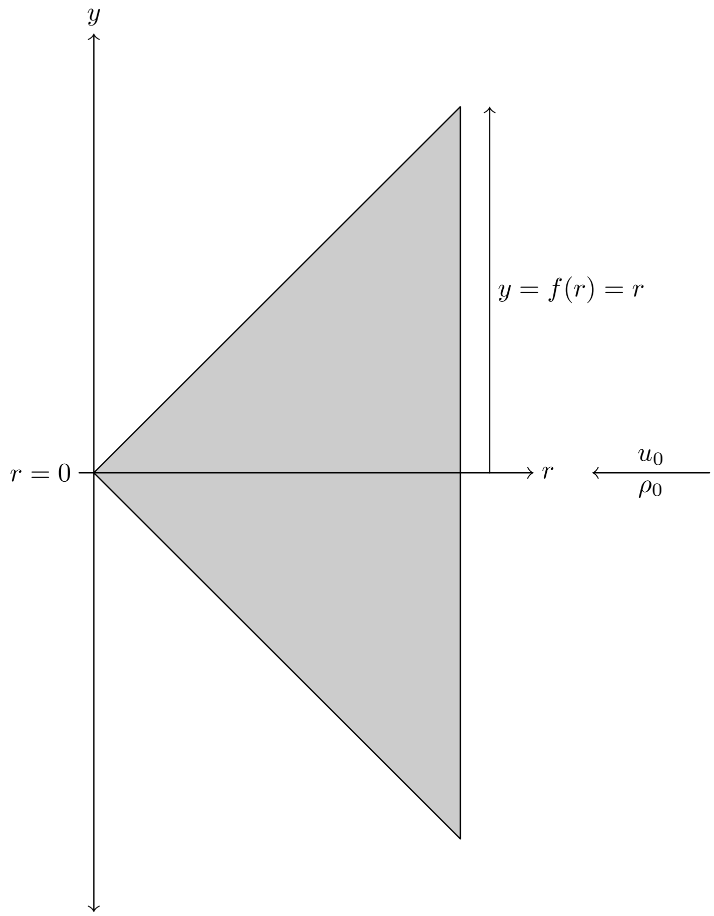

FIGURE 2. Noh problem setup at $t = 0$ in a conical channel.

4.2. **Auxiliary conditions and the group generator.** The first step to solving this problem using symmetry methods is to determine the admissible values for the group parameters permitted under the auxiliary conditions of the Noh problem. First, the initial conditions $F_4 = 0$ and $F_5 = 0$ must satisfy Equation (3.20) where

$$F_4 = u - u_0 = 0, \tag{4.5}$$

$$F_5 = \rho - \rho_0 = 0, \tag{4.6}$$

and u_0 and ρ_0 are constants. Substituting Equation (4.5) into Equation (3.20)

$$\mathrm{pr}^{(1)}V\ (u - u_0)|_{u=u_0} = 0, \tag{4.7}$$

which leads to

$$a_4 u|_{u=u_0} = a_4 u_0 = (a_1 - a_2)u_0 = 0, \tag{4.8}$$

and therefore

$$a_1 = a_2, \tag{4.9}$$

for a non-trivial initial velocity. Repeating for Equation (4.6)

$$\mathrm{pr}^{(1)}V\ (\rho - \rho_0)|_{\rho=f(r,t)} = a_3 \rho|_{\rho=\rho_0} = 0, \tag{4.10}$$

and therefore

$$a_3 = 0 \tag{4.11}$$

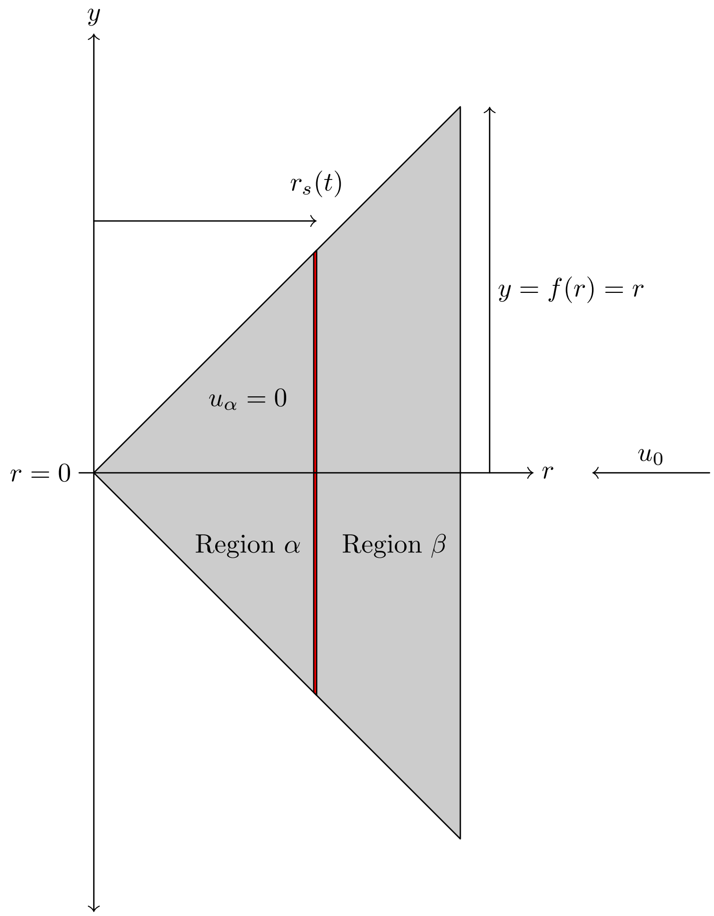

FIGURE 3. The Noh problem at $t > 0$ in a conical channel.

for a non-trivial initial density. Under these restrictions, the infinitesimal group generator simply consists of a scaling in space and time

$$V_{\text{Noh}} = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t}. \tag{4.12}$$

Ensuring that Equations (4.1) – (4.3) also satisfy Equation (3.20) further requires the shock velocity to be constant

$$D = \text{constant}. \tag{4.13}$$

By Equation (3.42) the restrictions from the auxiliary equations yield the similarity variable

$$\xi = \frac{r}{t}. \tag{4.14}$$

Similarly, by inspection of Equations (3.43) – (3.45), the density, velocity and pressure are simply unknown functions of ξ

$$\rho = \hat{\rho}(\xi), \tag{4.15}$$

$$u = \hat{u}(\xi), \tag{4.16}$$

$$p = \hat{p}(\xi), \tag{4.17}$$

and the reduced channel flow system, Equations (3.48) – (3.50), becomes

$$A\left[-\xi\frac{\partial\hat{\rho}}{\partial\xi}+\frac{\partial(\hat{\rho}\hat{u})}{\partial\xi}\right]+A_t t\hat{\rho}+A_r t\hat{\rho}\hat{u}=0, \tag{4.18}$$

$$-\xi\hat{\rho}\frac{\partial\hat{u}}{\partial\xi}+\frac{\partial\hat{p}}{\partial\xi}+\hat{\rho}\hat{u}\frac{\partial\hat{u}}{\partial\xi}=0, \tag{4.19}$$

$$A\left[\hat{\rho}e_{\hat{p}}\left(\hat{u}\frac{\partial\hat{p}}{\partial\xi}-\xi\frac{\partial\hat{p}}{\partial\xi}\right)+\hat{\rho}e_{\hat{\rho}}\left(\hat{u}\frac{\partial\hat{\rho}}{\partial\xi}-\xi\frac{\partial\hat{\rho}}{\partial\xi}\right)+\hat{p}\frac{\partial\hat{u}}{\partial\xi}\right]+A_t t\hat{p}+A_r t\hat{p}\hat{u}=0. \tag{4.20}$$

4.3. **A cross-sectional area function and the isentropic bulk modulus.** In order to further progress towards a solution, a function for the area $A := A(r,t)$ must be specified. In Section 3.1 it was determined that an area function of the form given by Equation (3.38) requires $a_1 = a_2$, thus this choice is compatible with the initial conditions of the Noh problem. Substituting $A = c_1 r^{c_2} + c_3 t^{c_2}$ and therefore $A_r = c_1 c_2 r^{c_2-1}$ and $A_t = c_2 c_3 t^{c_2-1}$ into Equations (4.18) – (4.20)

$$-\xi\hat{\rho}'+\hat{u}\hat{\rho}'+\hat{\rho}\left(\hat{u}'+\frac{c_2c_3+c_1c_2\xi^{c_2-1}\hat{u}}{c_1\xi^{c_2}+c_3}\right)=0, \tag{4.21}$$

$$-\xi\hat{u}'+\hat{u}\hat{u}'+\frac{\hat{p}'}{\hat{\rho}}=0, \tag{4.22}$$

$$\hat{\rho}e_{\hat{p}}\left(\hat{u}\hat{p}'-\xi\hat{p}'\right)+\hat{\rho}e_{\hat{\rho}}\left(\hat{u}\hat{\rho}'-\xi\hat{\rho}'\right)+\hat{p}\left(\hat{u}'+\frac{c_2c_3+c_1c_2\xi^{c_2-1}\hat{u}}{c_1\xi^{c_2}+c_3}\right)=0, \tag{4.23}$$

where the primes denote differentiation with respect to ξ and we have assumed $c_1 \neq 0$. Next, dividing Equation (4.23) through by $\hat{\rho}e_{\hat{p}}$ gives

$$-\xi\hat{p}'+\hat{u}\hat{p}'+\frac{1}{e_{\hat{p}}}\left[e_{\hat{\rho}}\left(\hat{u}\hat{\rho}'-\xi\hat{\rho}'\right)+\frac{\hat{p}}{\hat{\rho}}\left(\hat{u}'+\frac{c_2c_3+c_1c_2\xi^{c_2-1}\hat{u}}{c_1\xi^{c_2}+c_3}\right)\right]=0. \tag{4.24}$$

From Equation (4.21)

$$\hat{u}\hat{\rho}'-\xi\hat{\rho}'=-\hat{\rho}\left(\hat{u}'+\frac{c_2c_3+c_1c_2\xi^{c_2-1}\hat{u}}{c_1\xi^{c_2}+c_3}\right).$$

Substituting this result into Equation (4.24) and rearranging gives

$$-\xi\hat{p}'+\hat{u}\hat{p}'+\frac{1}{e_{\hat{p}}}\left(\frac{\hat{p}}{\hat{\rho}}-\hat{\rho}e_{\hat{\rho}}\right)\left(\hat{u}'+\frac{c_2c_3+c_1c_2\xi^{c_2-1}\hat{u}}{c_1\xi^{c_2}+c_3}\right)=0. \tag{4.25}$$

Identifying the coefficient of the third term as being equivalent to the isentropic bulk modulus K

$$K=-\rho\frac{s_\rho}{s_p}=\frac{1}{e_p}\left(\frac{p}{\rho}-\rho e_\rho\right), \tag{4.26}$$

where s denotes the entropy as a function of density and pressure $s := s(\rho,p)$, (see Axford [3, p. 2, eq. 7] for details) we arrive at the following reduced system of channel flow equations

$$-\xi\hat{\rho}'+\hat{u}\hat{\rho}'+\hat{\rho}\left(\hat{u}'+\frac{c_2c_3+c_1c_2\xi^{c_2-1}\hat{u}}{c_1\xi^{c_2}+c_3}\right)=0, \tag{4.27}$$

$$-\xi\hat{u}'+\hat{u}\hat{u}'+\frac{\hat{p}'}{\hat{\rho}}=0, \tag{4.28}$$

$$-\xi\hat{p}'+\hat{u}\hat{p}'+K\left(\hat{u}'+\frac{c_2c_3+c_1c_2\xi^{c_2-1}\hat{u}}{c_1\xi^{c_2}+c_3}\right)=0. \tag{4.29}$$

For $c_3 = 0$ and c_2 set to discrete values of 0,1 or 2, this system of equations is analogous to that presented in Ramsey et al. [33].

4.4. **Solution in the unshocked region.** Given the problem set up described in Section 4.1, for some time $t > 0$ the velocity of the flow into the channel remains constant and is therefore equivalent to the initial velocity prescribed at $t = 0$, $\hat{u} = u_0 < 0$. Under this condition, the conservation equations, Equations (4.27)–(4.29), become

$$(u_0 - \xi)\,\hat{\rho}' + \hat{\rho}\left(\frac{c_2c_3 + c_1c_2\xi^{c_2-1}u_0}{c_1\xi^{c_2} + c_3}\right) = 0, \tag{4.30}$$

$$\frac{\hat{p}'}{\hat{\rho}} = 0, \tag{4.31}$$

$$-\xi\hat{p}' + u_0\hat{p}' + K\left(\frac{c_2c_3 + c_1c_2\xi^{c_2-1}u_0}{c_1\xi^{c_2} + c_3}\right) = 0. \tag{4.32}$$

Equation (4.30) is satisfied for

$$\hat{\rho}_\beta = \nu\frac{(\xi - u_0)^{c_2}}{c_1\xi^{c_2} + c_3}, \tag{4.33}$$

$$= \nu\frac{(r - u_0t)^{c_2}}{c_1r^{c_2} + c_3t^{c_2}}, \tag{4.34}$$

where ν is the integration constant. Applying the initial condition $\rho_0 = \hat{\rho}(r, t = 0)$ gives

$$\hat{\rho}_\beta = \begin{cases} \rho_0\dfrac{c_1(r - u_0t)^{c_2}}{c_1r^{c_2} + c_3t^{c_2}}, & c_2 \neq 0, \\ \rho_0, & c_2 = 0. \end{cases} \tag{4.35}$$

For $c_3 = 0$ this result again agrees with that provided in [33, Eqn. 24].

Given this result for $\hat{\rho}_\beta$, Equation (4.31) is satisfied for

$$\hat{p}_\beta = \text{constant}. \tag{4.36}$$

Thus, the solutions to Equation (4.32) are determined by solving

$$K\left(\frac{c_2c_3 + c_1c_2\xi^{c_2-1}u_0}{c_1\xi^{c_2} + c_3}\right) = 0. \tag{4.37}$$

Recalling the previous assumptions, $c_1 \neq 0$ and $u_0 < 0$, this equation is satisfied in two cases

$$K = 0, \qquad \text{or} \qquad c_2 = 0. \tag{4.38}$$

Substituting $K = 0$ into Equation (4.26) for the isentropic bulk modulus yields

$$\frac{\partial\hat{e}}{\partial\hat{\rho}} = \frac{\hat{p}}{\hat{\rho}^2}. \tag{4.39}$$

Holding $\hat{p}$ as constant and integrating

$$\hat{e} + \frac{\hat{p}}{\hat{\rho}} = f(\hat{p}), \tag{4.40}$$

where f is an arbitrary function of its argument. Equation (4.40) is satisfied for either

$$\hat{\rho}_\beta = \text{constant}, \qquad \text{or} \qquad \hat{e}_\beta + \frac{\hat{p}_\beta}{\hat{\rho}_\beta} = \text{constant}. \tag{4.41}$$

Since, for the unshocked region, it was determined that the solution for the density profile, Equation (4.35), is not constant, the first case of Equation (4.41) is ruled out. Consequently, substituting the solution for the density and pressure profiles for the unshocked region into the latter case yields the following equality that must be satisfied

$$e_\beta + \frac{p_\beta}{\rho_0}\frac{(c_1r^{c_2} + c_3t^{c_2})}{c_1(r - u_0t)^{c_2}} = \text{constant}. \tag{4.42}$$

4.5. **Solution in the shocked region.** At some time $t > 0$, the velocity of the flow in the shocked region is zero, that is, $\hat{u}_\alpha = 0$. Thus, Equations (4.27)–(4.29) become

$$-\xi\hat{\rho}' + \hat{\rho}\frac{c_2 c_3}{c_1\xi^{c_2} + c_3} = 0, \tag{4.43}$$

$$\frac{\hat{p}'}{\hat{\rho}} = 0, \tag{4.44}$$

$$-\xi\hat{p}' + K\frac{c_2 c_3}{c_1\xi^{c_2} + c_3} = 0. \tag{4.45}$$

Equation (4.43) is satisfied when

$$\hat{\rho}_\alpha = \nu\frac{\xi^{c_2}}{c_1\xi^{c_2} + c_3}, \tag{4.46}$$

$$= \nu\frac{r^{c_2}}{c_1 r^{c_2} + c_3 t^{c_2}}, \tag{4.47}$$

where ν is the integration constant. When $c_3 = 0$, $\rho_\alpha =$ constant which agrees with the result provided in [33].

Equation (4.44) is satisfied by

$$\hat{p}_\alpha = \text{constant}. \tag{4.48}$$

Noting this, Equation (4.45) becomes

$$K\frac{c_2 c_3}{c_1\xi^{c_2} + c_3} = 0, \tag{4.49}$$

which is satisfied when either $K = 0$, $c_2 = 0$ or $c_3 = 0$. Solving the $K = 0$ case as before yields the result given in Equation (4.40) which for the shocked region is satisfied for either

$$\hat{\rho}_\alpha = \text{constant}, \qquad \hat{e}_\alpha + \frac{\hat{p}_\alpha}{\hat{\rho}_\alpha} = \text{constant}. \tag{4.50}$$

If c_2 and c_3 are simultaneously non-zero, Equation (4.47) yields a non-constant density and therefore the latter case in (4.50) must be satisfied.

To summarize, for curvilinear geometries, i.e., where $c_2 \neq 0$, the Noh problem imposes constraints on the EOS for solutions to exist. In the case of a time independent cross-sectional area, it is necessary for the EOS to satisfy only Equation (4.42). However, by including time dependence, i.e., $c_3 \neq 0$, the latter equality in Equation (4.50) must additionally be satisfied.

4.6. **A solution using the ideal gas EOS.** Up to this point, the solutions in the unshocked and shocked regions defined in Sections 4.4 and 4.5, respectively have been determined without defining a specific EOS. For planar geometries, it was determined that no restrictions on the EOS are required for a solution of the Noh problem to exist. However, for $c_2 \neq 0$ and depending on the value of c_3, either Equation (4.42) or both Equation (4.42) and the latter equality in Equation (4.50) must be satisfied. For simplicity and example purposes, the following discussion provides a solution pertaining to the ideal gas EOS since it supports the existence of a Noh solution in all geometries. Solutions to the Noh problem for a variety of EOS models including the ideal gas case have previously been presented by Axford [3] and generalized by Ramsey et al. [33]. By modelling the fluid flow using the quasi-1D model, further generalization of these results has been achieved including the addition of time dependent behavior for the cross-section and extension of the channel geometries considered.

4.6.1. $K \neq 0$. First, the case where the isentropic bulk modulus $K \neq 0$ is explored. By inspection of Equation (4.37), in order for the energy equation to be satisfied in the unshocked region, we require $c_2 = 0$. Under this restriction, the energy equation in the shocked region is simultaneously satisfied for non-zero K and by Equation (4.35) the density in the unshocked region is constant

$$\rho_\beta = \rho_0. \tag{4.51}$$

Combining the constant unshocked density with the velocities in the shocked and unshocked regions and the ideal gas EOS

$$u_\alpha = 0, \qquad u_\beta = u_0, \qquad e(\rho, p) = \frac{p}{(\gamma - 1)\rho}, \tag{4.52}$$

the jump conditions reduce to

$$\rho_\alpha = \rho_0 \left(1 - \frac{u_0}{D}\right), \tag{4.53}$$

$$p_\beta - p_\alpha = \rho_0 (D - u_0) u_0, \tag{4.54}$$

$$e_\beta - e_\alpha = \frac{p_\beta u_0}{\rho_0 (D - u_0)} - \frac{1}{2} u_0^2 = \frac{p_\beta}{(\gamma - 1)\rho_0} - \frac{p_\alpha}{(\gamma - 1)\rho_\alpha}. \tag{4.55}$$

Solving for the shockwave velocity and then back substituting into Equations (4.53) and (4.54), the solution is

$$D = -\frac{u_0(\gamma - 3)}{4} + \sqrt{\frac{p_\beta \gamma}{\rho_0} + \frac{u_0^2(\gamma + 1)^2}{16}}, \tag{4.56}$$

$$\rho_\alpha = \rho_0 \left(1 + \frac{u_0}{\dfrac{u_0(\gamma - 3)}{4} - \sqrt{\dfrac{p_\beta \gamma}{\rho_0} + \dfrac{u_0^2(\gamma + 1)^2}{16}}}\right), \tag{4.57}$$

$$p_\alpha = p_\beta - \rho_0 u_0 \left(-\frac{u_0(\gamma + 1)}{4} + \sqrt{\frac{p_\beta \gamma}{\rho_0} + \frac{u_0^2(\gamma + 1)^2}{16}}\right). \tag{4.58}$$

4.6.2. $K = 0$. Next, the case where $K = 0$ is considered which, by inspection of Equations (4.37) and (4.49), allows for arbitrary values of c_2 and c_3. For the subcase $K = c_3 = 0$, Equation (4.50) is satisfied since $\hat{\rho}_\alpha =$ constant. However, for the unshocked region, $\rho_\beta \neq$ constant. Therefore if $K = 0$, the EOS must satisfy Equation (4.42). For the ideal gas EOS, this leads to

$$\frac{\gamma p_\beta}{(\gamma - 1)\rho_0} \left(\frac{r}{r - u_0 t}\right)^{c_2} = \text{constant}, \tag{4.59}$$

which requires $p_\beta = 0$. Rearranging Equation (4.2) for the shock velocity and substituting $p_\beta = 0$ yields

$$D = -\frac{p_\alpha}{u_0 \rho_\alpha}. \tag{4.60}$$

Since p_α, u_0 and ρ_α are all constant, D must also be constant. Given this result and Equation (4.4) we can integrate with respect to time to obtain

$$\int_0^t D \mathrm{d}t = \int_0^t \frac{\mathrm{d}r_s(t)}{\mathrm{d}t} \mathrm{d}t, \tag{4.61}$$

which results in

$$Dt = r_s(t) - r_s(0), \tag{4.62}$$

$$= r_s(t). \tag{4.63}$$

This relationship defines the spatial coordinate r in terms of the shock velocity and time

$$r_s = Dt. \tag{4.64}$$

Next, by the following equalities

$$p_\beta = 0, \qquad \frac{r_s}{t} = D, \qquad u_\alpha = 0, \qquad u_\beta = 0, \qquad \rho_\beta = \rho_0\left(1 - \frac{u_0 t}{r}\right)^{c_2},$$

the jump equations become

$$\rho_\alpha = \rho_0\left(1 - \frac{u_0}{D}\right)^{c_2+1}, \tag{4.65}$$

$$p_\alpha = -\rho_\alpha D u_0, \tag{4.66}$$

$$e_\beta - e_\alpha = \frac{1}{2}u_0^2 = \frac{p_\alpha}{(\gamma-1)\rho_\alpha}. \tag{4.67}$$

Again, solving for the shock velocity and back substituting into Equations (4.65) – (4.67) the solution is

$$D = -\frac{u_0(\gamma-1)}{2}, \tag{4.68}$$

$$\rho_\alpha = \rho_0\left(\frac{\gamma+1}{\gamma-1}\right)^{c_2+1}, \tag{4.69}$$

$$p_\alpha = \rho_0\left(\frac{\gamma+1}{\gamma-1}\right)^{c_2+1}\frac{u_0^2}{2(\gamma-1)}. \tag{4.70}$$

Finally, for the case where $K = 0$ but $c_3 \neq 0$ the density profile in the shocked region is no longer constant unless $c_2 = 0$. Therefore the latter case of Equation (4.50) must also be satisfied. For the ideal gas, satisfying Equation (4.42) leads to the same result as before, namely $p_\beta = 0$. Satisfying Equation (4.50) however requires

$$\frac{\gamma p_\alpha}{(\gamma-1)\rho_\alpha} = \text{constant}, \tag{4.71}$$

which further requires $p_\alpha = 0$. Substituting $p_\alpha = p_\beta = 0$ into Equation (4.2) yields a zero shock velocity $D = 0$. Since there is no shock generated there is no α-region created. Thus, the solution for the shock velocity, pressure and density within the channel is simply

$$D = 0, \tag{4.72}$$

$$\rho_\beta = \rho_0\frac{c_1(r-u_0t)^{c_2}}{c_1r^{c_2}+c_3t^{c_2}}, \qquad c_2 \neq 0 \tag{4.73}$$

$$p_\beta = 0. \tag{4.74}$$

Physically, this solution corresponds to the channel walls moving at a rate in time that mitigates the production of a shockwave.

5. Conclusions

In summary, the equations pertaining to quasi-1D channel flow were studied within the context of symmetry analysis. As a result, we identified a coupling between the parameters determining the scaling transformations, the cross-section of the channel and the material properties of the fluid i.e., the EOS. Following these results, various admissible combinations of scaling parameters, cross-sectional area functions and EOS models were discussed. In particular, for the case of a time-independent cross-section, it was determined that the most general function for the area permitting arbitrary values for a_1 and a_2 was given by $A = c_1r^{c_2}$ where c_1 and c_2 are constants. Only if $a_1 = 0$ are scaling solutions possible for more general time-independent functions for the channel cross-section.

By making the assumption that the channel flow was quasi-1D, the values of c_1 and c_2 appearing in the cross-section $A = c_1 r^{c_2}$ were permitted to vary continuously while still permitting scaling symmetry. This result is an extension of the known scaling symmetries admitted for exact 1D flow corresponding to $c_2 = 0, 1$ or 2. The symmetry analysis was then further extended by the addition of time dependence to the channel cross-section. By applying the symmetry analysis results, the quasi-1D Noh problem was analysed for a channel cross-section that was compatible with the symmetry contraints imposed by the problem initial conditions. A conditional solution was determined for an unspecified EOS. This conditional solution was subsequently evaluated to obtain an explicit solution for the case of an ideal gas.

There are many possibilities for further extension of this work. One might involve the extension of other existing solutions for 1D self-similar problems via the quasi-1D assumption. Examples could be the Sedov-Taylor-von Neuman point explosion or converging Guderley problems. Solutions to these problems were obtained using Lie groups of scaling transformations. However, previous work by Ovsiannikov [32], and many others [2, 8, 13, 14] have demonstrated that the Euler equations governing compressible, inviscid flow admit additional symmetry transformations. Included are translation, Galilean and projective transformations in addition to the numerous possible linear combinations of these elementary symmetries. At present, it is unknown how the cross-section of the channel is coupled to these additional symmetries and whether the 8−parameter group reported by Ovsiannikov can be extended by the additional flexibility allowed in the cross-section for quasi-1D flow.

Further work might also extend the results given here for point symmetries to generalized symmetries of the kind discussed in Olver [31, Chap. 5] or even weak symmetries which are discussed in Olver and Rosenau [30]. Generalized symmetries add a level of complexity compared to point symmetries by allowing the transformations to depend on derivatives in addition to the independent and dependent variables.

Alternatively, future work could pursue the case where $a_1 = 0$. In the analysis given, it was assumed $a_1 \neq 0$ which restricted the channel geometries permitted. If $a_1 = 0$, a broader set of geometries may be considered however the number of initial/boundary conditions admitted by such a restricted group generator are limited. This in turn limits the shockwave problems that can be addressed. Consequently, the focus of such work could be directed towards constructing shock-less solutions similar to those given in Coggeshall [13] and McHardy et al. [28].

Lastly, other extensions of the current work could include the addition of viscosity or heat conduction models or both to the governing equations to understand how these alter the symmetry groups admitted.

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Appendix A. The channel flow equations: Derivation

We begin by stating the integral equations of motion for a *material volume* [41, Chap 1, p.32-33]. Here the material volume refers to the volume of a parcel of fluid that moves at the fluid velocity u, undergoing deformation with no mass flux through its encompassing boundary. Consequently, the time rate of change of mass within a material volume is zero

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^*(t)} \rho \mathrm{d}V = 0, \tag{A.1}$$

where $\partial\Omega^*(t)$ denotes the material volume surface. Next, the momentum balance is defined by equating the time rate of change of the material volume momentum to the net sum of the body forces plus the surface forces

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^*(t)}\rho\boldsymbol{u}\mathrm{d}V = \underbrace{\int_{\Omega^*(t)}\rho\boldsymbol{G}\mathrm{d}V}_{\text{body forces}} + \underbrace{\int_{\partial\Omega^*(t)}\boldsymbol{T}\mathrm{d}S}_{\text{surface forces}}, \tag{A.2}$$

where $\boldsymbol{G}$ is the specific body force vector causing the material volume to accelerate and $\boldsymbol{T}$ is the surface traction vector which is simply a surface force per unit area. For the purposes of this paper, the effect of any viscous forces are neglected and therefore the viscous stress tensor is zero. As a result, the only contributions to the surface traction vector enter through hydrostatic pressure forces, i.e., the traction vector simply corresponds to the fluid pressure force acting normal to the surface

$$\boldsymbol{T} = -p\boldsymbol{n}, \tag{A.3}$$

where $\boldsymbol{n}$ is the outward surface unit normal vector. Finally, the energy balance is constructed by relating the time rate of change of energy, E, to the energy transfer rate generated via the mechanisms of heat and work

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^*(t)}\rho E\mathrm{d}V = \underbrace{\int_{\Omega^*(t)}\rho\boldsymbol{G}\cdot\boldsymbol{u}\mathrm{d}V + \int_{\partial\Omega^*(t)}\boldsymbol{T}\cdot\boldsymbol{u}\mathrm{d}S}_{\text{work}} - \underbrace{\int_{\partial\Omega^*(t)}\boldsymbol{q}\cdot\boldsymbol{n}\mathrm{d}S}_{\textit{heat}}, \tag{A.4}$$

where $E = e + u^2/2$. The two components of the energy considered are the specific internal and the specific kinetic energies denoted e and $u^2/2$, respectively. In the subsequent discussion, as well as neglecting any viscous effects, all body forces, such as gravitational forces, are also ignored. Furthermore, the heat conduction terms in the energy equation are excluded. As a result, there are no mechanisms included to facilitate the increase of entropy within the system. It is recognized that these effects play a significant role near the channel walls and that the approximation will deteriorate as the length of the channel is increased. The applicability of the analysis is therefore restricted to reasonably short channels. For an idea of what constitutes "short" the reader is referred to Thompson [41, Chap. 6] where frictional forces are discussed. In addition to the assumption that the channel is "short", the success of the model is also dependent on the channel geometry. With the exception of the geometries corresponding to the exact cases considered in [33], the most successful results are obtained when

$$\frac{\mathrm{d}a}{\mathrm{d}r} << 1 \qquad \text{and} \qquad \frac{a}{C} << 1, \tag{A.5}$$

where a denotes a measure of the cross-sectional area, e.g. the radius of the channel, and r and C denote the length along the channel and the radius of curvature of the channel wall, respectively.

Next, Equations (A.1) – (A.4) are applied to the *control volume* outlined in Figure 1. In order to achieve this we make use of the Reynold's transport theorem which states

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\chi\mathrm{d}V = \int_{\Omega(t)}\frac{\partial\chi}{\partial t}\mathrm{d}V + \int_{\partial\Omega(t)}\chi\boldsymbol{b}\cdot\boldsymbol{n}\mathrm{d}S, \tag{A.6}$$

where $\chi := \chi(r,t)$ is any summable, continuous function, $\Omega(t)$ denotes an arbitrary volume, b is the velocity at which the volume is moving relative to the observer in the stationary, laboratory reference frame and $\boldsymbol{n}$ is the outward unit normal to the surface $\partial\Omega$. Applying this to Equation (A.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^*(t)}\rho\mathrm{d}V = \int_{\Omega^*(t)}\frac{\partial\rho}{\partial t}\mathrm{d}V + \int_{\partial\Omega^*(t)}\rho\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}S = 0. \tag{A.7}$$

The Reynold's transport theorem can similarly be applied to the control volume which will now also be denoted by $\Omega(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho \mathrm{d}V = \int_{\Omega(t)}\frac{\partial\rho}{\partial t}\mathrm{d}V + \int_{\partial\Omega(t)}\rho\boldsymbol{b}\cdot\boldsymbol{n}\mathrm{d}S, \tag{A.8}$$

where b is now the velocity of the control volume. Next, taking the arbitrary material volume discussed previously and matching it exactly to the control volume so that

$$\Omega(t) = \Omega^*(t), \qquad \partial\Omega(t) = \partial\Omega^*(t), \tag{A.9}$$

Equation (A.7) can be substituted into Equation (A.8) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho \mathrm{d}V + \int_{\partial\Omega(t)}\rho(\boldsymbol{u}-\boldsymbol{b})\cdot\boldsymbol{n}\mathrm{d}S = 0. \tag{A.10}$$

Finally, by fixing the control volume to be stationary, $b = 0$, this result reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho \mathrm{d}V + \underbrace{\int_{\partial\Omega(t)}\rho(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S}_{\text{mass flow in/out}} = 0. \tag{A.11}$$

The momentum and energy equations for the control volume can be found analogously. By neglecting the body forces, viscosity and heat conduction, the resulting equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho\boldsymbol{u} \mathrm{d}V + \underbrace{\int_{\partial\Omega(t)}\rho\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S}_{\text{momentum flow in/out}} = \int_{\partial\Omega(t)}\boldsymbol{T}\mathrm{d}S, \tag{A.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho E \mathrm{d}V + \underbrace{\int_{\partial\Omega(t)}\rho E(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S}_{\text{energy flow in/out}} = \int_{\partial\Omega(t)}\boldsymbol{T}\cdot\boldsymbol{u}\mathrm{d}S, \tag{A.13}$$

A.1. **Mass.** Starting from the control volume continuity equation, Equation (A.11), under the assumption that there is no fluid flux through the channel walls, i.e., the only flow into and out of the control volume is through the permeable left and right end surface areas, $\partial\Omega_1$ and $\partial\Omega_2$, respectively, the second integral term can be expressed simply as the sum of two components

$$\int_{\partial\Omega(t)}\rho\boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}S = \int_{\partial\Omega_1(t)}\rho\boldsymbol{u}\cdot\boldsymbol{n_1}\mathrm{d}S + \int_{\partial\Omega_2(t)}\rho\boldsymbol{u}\cdot\boldsymbol{n_2}\mathrm{d}S, \tag{A.14}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are unit normal vectors to surfaces, $\partial\Omega_1$ and $\partial\Omega_2$. The control volume defined in Figure 1 has left and right surfaces corresponding to cross-sectional areas of the channel. Consequently, the unit normals of these surfaces are parallel to the channel axis which points in the direction $\hat{\imath}$. The unit normal vectors are therefore given by

$$\boldsymbol{n}_1 = -\hat{\imath}, \qquad \boldsymbol{n}_2 = \hat{\imath}. \tag{A.15}$$

Next, evaluating the dot products in Equation (A.14) gives

$$\boldsymbol{u}\cdot\boldsymbol{n}_1 = -u_i, \qquad \boldsymbol{u}\cdot\boldsymbol{n}_2 = u_i, \tag{A.16}$$

where u_i denotes the magnitude of the axial component of the velocity vector. Next, the flows are approximated to be quasi-1d. This assumption enables multi-dimensional flows to be treated using a simpler 1-d model. In real flows, the distributions of the pressure, density and magnitude of the flow velocity components parallel to the axis of the channel vary over the channel cross sectional area. In the quasi-1d assumption, all axial velocity components, pressures and densities assume the mean value across the channel cross section. Any transverse components of the velocities ensuring the flow remains in contact with the channel walls are only taken into account through conservation of mass. As a result, under the quasi-1d assumption, u_i corresponds to the mean value of the axial velocity component averaged over the cross-sectional area. In the subsequent discussion, the subscript i is dropped and u_i is denoted simply by u. Equation (A.11) therefore becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho \mathrm{d}V + \int_{\partial\Omega_2(t)}\rho u \mathrm{d}S - \int_{\partial\Omega_1(t)}\rho u \mathrm{d}S = 0. \tag{A.17}$$

The second and third integrals appearing can be evaluated explicitly since the density and velocity no longer vary over the cross-section giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho \mathrm{d}V + (\rho u A)|_{r_2} - (\rho u A)|_{r_1} = 0, \tag{A.18}$$

where r_1 and r_2 denote the spatial locations of the surfaces $\partial\Omega_1$ and $\partial\Omega_2$ respectively and $A(r,t)$ denotes the cross-sectional area of the channel.

Next, the remaining volume integration can be separated into integrals over the cross sectional area function $A(r,t)$ and the length of the control volume

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{r_1}^{r_2}\int_{0}^{A(r,t)}\rho \mathrm{d}S\mathrm{d}r = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta r}\rho A \mathrm{d}r, \tag{A.19}$$

where the area integral has again been evaluated since the density is assumed constant over the cross section. Finally, substituting this result into Equation (A.18), dividing the whole equation through by Δr and taking the limit as $\Delta r \to 0$

$$\lim_{\Delta r\to 0}\frac{1}{\Delta r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta r}\rho A \mathrm{d}r + \frac{(\rho u A)|_{r_2} - (\rho u A)|_{r_1}}{\Delta r} = 0, \tag{A.20}$$

yields the channel flow continuity equation

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho u A)}{\partial r} = 0. \tag{A.21}$$

A.2. **Momentum.** Beginning with Equation (A.12), the channel flow momentum equation is derived by following an approach similar to the derivation of the mass equation. First, the surface traction vector is replaced using Equation (A.3)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho \boldsymbol{u} \mathrm{d}V + \underbrace{\int_{\partial\Omega(t)}\rho \boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S}_{\text{momentum flow in/out}} = -\int_{\partial\Omega(t)} p\boldsymbol{n}\mathrm{d}S. \tag{A.22}$$

Next, we only consider contributions pertaining to momentum changes in the direction $\hat{\imath}$ along the channel axis and therefore the momentum equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho(\hat{\imath}\cdot\boldsymbol{u})\mathrm{d}V + \int_{\partial\Omega(t)}\rho(\hat{\imath}\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S = -\int_{\partial\Omega(t)} p(\hat{\imath}\cdot\boldsymbol{n})\mathrm{d}S. \tag{A.23}$$

The dot product $\hat{\imath}\cdot\boldsymbol{u}$ equals u_i which in the quasi-1d approximation becomes the mean value u at all points over the cross-section. Once again, the volume integral on the left hand side of the equality can be separated into an integral over the channel cross-section and the length of the channel

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)}\rho u \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{r_1}^{r_2}\int_{0}^{A(r,t)}\rho u \mathrm{d}S\mathrm{d}r = \int_{\Delta r}\rho A \mathrm{d}r. \tag{A.24}$$

Also, as was true for the mass flow, the flow of momentum out and in only takes place through the surface areas $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Consequently, the second integral of Equation (A.23) is reduced to a sum of two components

$$\int_{\partial\Omega(t)}\rho u(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S = \int_{\partial\Omega_1(t)}\rho u(\boldsymbol{u}\cdot\boldsymbol{n_1})\mathrm{d}S + \int_{\partial\Omega_2(t)}\rho u(\boldsymbol{u}\cdot\boldsymbol{n_2})\mathrm{d}S, \tag{A.25}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are again the unit normal vectors to the surfaces $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Evaluating the dot products as we did for the mass equation gives

$$\int_{\partial\Omega(t)} \rho u(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S = \int_{\partial\Omega_2(t)} \rho u^2 \mathrm{d}S - \int_{\partial\Omega_1(t)} \rho u^2 \mathrm{d}S \tag{A.26}$$

$$= (\rho u^2 A)\big|_{r_2} - (\rho u^2 A)\big|_{r_1}. \tag{A.27}$$

Next, we evaluate the remaining integral on the right hand side (r.h.s.) of Equation (A.23). As a reminder, this term represents the surface forces in the $\hat{\imath}$ direction acting on the control volume. For the case of fluid flow through a channel in the absence of viscous effects, this term amounts to the forces imparted on the control volume by the channel walls as well as from the pressure of the fluid on either side of the control volume. Evaluating this term requires application of Gauss's divergence theorem. This theorem states the flux of a vector field through a closed surface is equivalent to the volume integral of the divergence of the vector field over the volume enclosed by the surface. It can be represented as

$$\oint_{\partial\Omega(t)} \boldsymbol{F}\cdot\boldsymbol{n}\mathrm{d}S = \int_{\Omega(t)} \nabla\cdot\boldsymbol{F}\mathrm{d}V, \tag{A.28}$$

where $\boldsymbol{F}$ denotes the vector field. Applying the divergence theorem to the r.h.s. integral of Equation (A.23) gives

$$-\int_{\partial\Omega(t)} p(\hat{\imath}\cdot\boldsymbol{n})\mathrm{d}S = -\int_{\Omega(t)} (\nabla\cdot p\hat{\imath})\mathrm{d}V = \int_{\Omega(t)} \frac{\partial p}{\partial r}\mathrm{dV}. \tag{A.29}$$

Splitting up the volume integral into integrals over the cross section and the length of the channel gives

$$-\int_{\partial\Omega(t)} p(\hat{\imath}\cdot\boldsymbol{n})\mathrm{d}S = -\int_{r_1}^{r_2}\int_0^{A(r,t)} \frac{\partial p}{\partial r}\mathrm{d}S\mathrm{d}r = -\int_{\Delta r} A\frac{\partial p}{\partial r}\mathrm{d}r, \tag{A.30}$$

where the surface integral has again been evaluated since the pressure does not vary over the cross section.

Using the results of Equations (A.24), (A.27) and (A.30), the momentum equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta r} \rho u A\mathrm{d}r + (\rho u^2 A)\big|_{r_2} - (\rho u^2 A)\big|_{r_1} + \int_{\Delta r} A\frac{\partial p}{\partial r}\mathrm{d}r = 0. \tag{A.31}$$

As before, dividing through by Δr and taking the limit as $\Delta r \to 0$

$$\lim_{\Delta r\to 0}\frac{1}{\Delta r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta r} \rho u A\mathrm{d}r + \frac{(\rho u^2 A)\big|_{r_2} - (\rho u^2 A)\big|_{r_1}}{\Delta r} + \frac{1}{\Delta r}\int_{\Delta r} A\frac{\partial p}{\partial r}\mathrm{d}r = 0, \tag{A.32}$$

gives

$$\frac{(\partial\rho A u)}{\partial t} + \frac{(\partial\rho A u^2)}{\partial r} + A\frac{\partial p}{\partial r} = 0. \tag{A.33}$$

which is equivalent to

$$\frac{\partial\left(\rho A u\right)}{\partial t} + \frac{\partial\left(\rho A u^2 + pA\right)}{\partial r} = p\frac{\partial A}{\partial r}. \tag{A.34}$$

A.3. **Energy.** Starting with Equation (A.13), the volume integral is reduced to an integral over the channel length

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega(t)} \rho E\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta r} \rho A(e + \frac{u^2}{2})\mathrm{d}r, \tag{A.35}$$

where the total energy has also been split into specific internal and specific kinetic energy components. Next, as was true in the mass and momentum equation derivations, the energy flow into and out of the control volume only occurs through the end surfaces. As such, the second integral in Equation (A.13) is again decomposed into two components

$$\int_{\partial\Omega(t)} \rho E(\boldsymbol{u}\cdot\boldsymbol{n})\mathrm{d}S = \int_{\partial\Omega_1(t)} \rho\left(e+\frac{u^2}{2}\right)(\boldsymbol{u}\cdot\boldsymbol{n_1})\mathrm{d}S + \int_{\partial\Omega_2(t)} \rho\left(e+\frac{u^2}{2}\right)(\boldsymbol{u}\cdot\boldsymbol{n_2})\mathrm{d}S, \tag{A.36}$$

$$= \left(\rho A u\left[e+\frac{u^2}{2}\right]\right)\bigg|_{r_2} - \left(\rho A u\left[e+\frac{u^2}{2}\right]\right)\bigg|_{r_1}. \tag{A.37}$$

Lastly, the final term appearing in Equation (A.13) can be evaluated again using the divergence theorem

$$\int_{\partial\Omega(t)} (\boldsymbol{T}\cdot\boldsymbol{u})\mathrm{d}S = -\int_{\partial\Omega(t)} p(\boldsymbol{n}\cdot\boldsymbol{u})\mathrm{d}S, \tag{A.38}$$

$$= -\int_{\Omega(t)} (\nabla\cdot p\boldsymbol{u})\mathrm{d}V, \tag{A.39}$$

$$= -\int_{\Delta r} A(\nabla\cdot p\boldsymbol{u})\mathrm{d}r. \tag{A.40}$$

Focusing on the divergence term appearing inside of the integral, in the quasi-1d approximation, since pressure only varies along the length of the channel, we find

$$A\left(\nabla\cdot p\boldsymbol{u}\right) = A\left[u\frac{\partial p}{\partial r} + p(\nabla\cdot\boldsymbol{u})\right], \tag{A.41}$$

which, by adding and subtracting terms, can be equivalently expressed as

$$A\left(\nabla\cdot p\boldsymbol{u}\right) = \frac{\partial(pAu)}{\partial r} - p\left[\frac{\partial(Au)}{\partial r} - A(\nabla\cdot\boldsymbol{u})\right]. \tag{A.42}$$

Next, the divergence of the velocity field is related to the material derivative of the density

$$\nabla\cdot\boldsymbol{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r}\right). \tag{A.43}$$

Substitution of this relationship into Equation (A.42) gives

$$A\left(\nabla\cdot p\boldsymbol{u}\right) = \frac{\partial(pAu)}{\partial r} - \frac{p}{\rho}\left[\rho\frac{\partial(Au)}{\partial r} + Au\frac{\partial\rho}{\partial r} + A\frac{\partial\rho}{\partial t}\right], \tag{A.44}$$

$$= \frac{\partial(pAu)}{\partial r} - \frac{p}{\rho}\left[\frac{\partial(\rho Au)}{\partial r} + A\frac{\partial\rho}{\partial t}\right]. \tag{A.45}$$

By Equation (A.21), the final term appearing in the square brackets of Equation (A.45) is equivalent to $-\rho\dfrac{\partial A}{\partial t}$ and therefore

$$A\left(\nabla\cdot p\boldsymbol{u}\right) = \frac{\partial(pAu)}{\partial r} + p\frac{\partial A}{\partial t}. \tag{A.46}$$

Combining the results of Equations (A.35), (A.37), (A.40) and (A.46), the energy balance equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Delta r} \rho A(e+\frac{u^2}{2})\mathrm{d}r + \left(\rho A u\left[e+\frac{u^2}{2}\right]\right)\bigg|_{r_2} - \left(\rho A u\left[e+\frac{u^2}{2}\right]\right)\bigg|_{r_1} + \int_{\Delta r}\left(\frac{\partial(pAu)}{\partial r} + p\frac{\partial A}{\partial t}\right)\mathrm{d}r = 0. \tag{A.47}$$

Dividing through by Δr once again and taking the limit as Δr goes to zero

$$\frac{\partial}{\partial t}\left(\rho A e + \frac{\rho A u^2}{2}\right) + \frac{\partial}{\partial r}\left(\rho A u e + \frac{\rho A u^3}{2} + pAu\right) = -p\frac{\partial A}{\partial t}. \tag{A.48}$$

Appendix B. Isentropic flow

Given the balance law for the energy contained within the channel control volume, it can be shown that this equation is analogous to Equation (2.3) which states that the flow is isentropic. First, the first and third terms in the energy equation of System (2.1) are expanded and regrouped to obtain

$$\rho A\underbrace{\left(\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial r}\right)}_{=\frac{\mathrm{d}e}{\mathrm{d}t}}+e\underbrace{\left(\frac{\partial(\rho A)}{\partial t}+\frac{\partial(\rho Au)}{\partial r}\right)}_{\text{mass }=0}+\frac{1}{2}\frac{\partial(\rho Au^2)}{\partial t}+\frac{1}{2}\frac{\partial\rho Au^3}{\partial r}+\frac{\partial pAu}{\partial r}+p\frac{\partial A}{\partial t}=0, \tag{B.1}$$

$$\Rightarrow \rho A\frac{\mathrm{d}e}{\mathrm{d}t}+\frac{1}{2}\frac{\partial(\rho Au^2)}{\partial t}+\frac{1}{2}\frac{\partial\rho Au^3}{\partial r}+\frac{\partial pAu}{\partial r}+p\frac{\partial A}{\partial t}=0. \tag{B.2}$$

Next, the second, third and fourth terms are similarly expanded and rearranged giving

$$\rho A\frac{\mathrm{d}e}{\mathrm{d}t}+\frac{u^2}{2}\underbrace{\left(\frac{\partial(\rho A)}{\partial t}+\frac{\partial(\rho Au)}{\partial r}\right)}_{\text{mass }=0}+Au\rho\underbrace{\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}\right)}_{\text{momentum }=0}+p\left(\frac{\partial(Au)}{\partial r}+\frac{\partial A}{\partial t}\right)=0, \tag{B.3}$$

$$\Rightarrow \rho A\frac{\mathrm{d}e}{\mathrm{d}t}+p\left(\frac{\partial(Au)}{\partial r}+\frac{\partial A}{\partial t}\right)=0. \tag{B.4}$$

From the Equation (A.21), it can be shown

$$-\frac{A}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}=\frac{\partial(Au)}{\partial r}+\frac{\partial A}{\partial t}. \tag{B.5}$$

Substituting this result above yields

$$\frac{\mathrm{d}e}{\mathrm{d}t}-\frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}=0. \tag{B.6}$$

From the Gibbs equation

$$\mathrm{d}e=T\mathrm{d}s+\frac{p}{\rho^2}\mathrm{d}\rho, \tag{B.7}$$

it is inferred

$$T\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{d}e}{\mathrm{d}t}-\frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}. \tag{B.8}$$

Substituting this final result into (B.6) results in

$$\frac{\mathrm{d}s}{\mathrm{d}t}=0. \tag{B.9}$$

Hence the flow is isentropic.

Appendix C. Solving the determining equations

The constraints between the group parameters, in addition to any further restrictions required for the admissibility of symmetry transformations can be determined by solving the set of determining equations, Equations (3.20). For example purposes, the first prolongation of the group generator is applied to the continuity equation appearing in System (2.1).

C.1. **Mass.** Starting with the continuity equation, expanding the partial derivatives and applying the first prolongation of the group generator specified in Equation (3.27) gives

$$\mathrm{pr}^{(1)}VF_1\Big|_{F_1=0} = 0 \Rightarrow \mathrm{pr}^{(1)}V\left(\frac{\partial\rho A}{\partial t} + \frac{\partial\rho Au}{\partial r}\right)\Big|_{F_1=0} = 0, \tag{C.1}$$

$$\begin{aligned} 0 = &[a_1 r\left(\rho_t A_r + \rho A_{tr} + \rho_r A_r u + \rho A_{rr} u + \rho A_r u_r\right),\\ &+ a_2 t\left(\rho_t A_t + \rho A_{tt} + \rho_r A_t u + \rho A_{rt} u + \rho A_t u_r\right),\\ &+ a_3\rho\left(A_t + A_r u + A u_r\right) + a_4 u\left(\rho_r A + \rho A_r\right),\\ &+ (a_3 - a_1)\rho_r\left(Au\right) + (a_3 - a_2)\rho_t\left(A\right) + (a_4 - a_1)u_r\left(\rho A\right)]|_{F_1=0}. \end{aligned}$$

At this point, the equation is factored in powers of ρ and the group parameters to get

$$\begin{aligned} 0 = \ &\rho[a_1(rA_{rt} + ruA_{rr} + ru_rA_r - u_rA)\\ &+ a_2(tA_{tt} + tuA_{rt} + tu_rA_t)\\ &+ a_3(A_t + uA_r + tu_rA_t)\\ &+ a_4(uA_r + u_rA)]\\ &+ [a_1(\rho_t rA_r + \rho_r ruA_r - \rho_r uA)\\ &+ a_2(\rho_t tA_t + \rho_r tuA_t - \rho_t A)\\ &+ a_3(\rho_r uA + \rho_t A)\\ &+ a_4\rho_r uA]. \end{aligned} \tag{C.2}$$

Next, the result must be evaluated at F_1=0 which from the continuity equation can be imposed by substituting for the density using

$$\rho = -\frac{A(\rho_t + u\rho_r)}{A_t + uA_r + u_rA}. \tag{C.3}$$

The substitution gives

$$\begin{aligned} 0 = &+ (A_t + uA_r + Au_r)[a_1(\rho_t rA_r + \rho_r ruA_r - \rho_r uA)\\ &\qquad + a_2(\rho_t tA_t + \rho_r tuA_t - \rho_t A)\\ &\qquad + a_3(\rho_r uA + \rho_t A)\\ &\qquad + a_4\rho_r uA],\\ &- A(\rho_t + u\rho_r)[a_1(rA_{rt} + ruA_{rr} + ru_rA_r - u_rA)\\ &\qquad + a_2(tA_{tt} + tuA_{rt} + tu_rA_t)\\ &\qquad + a_3(A_t + uA_r + tu_rA_t)\\ &\qquad + a_4(uA_r + u_rA)]. \end{aligned} \tag{C.4}$$

For the purposes of the symmetry analysis, the velocity u can be regarded as an independent variable. Consequently, Equation (C.4) can be treated as a polynomial in u. To be satisfied for all u, the coefficients of the polynomial must be zero. Separating the result accordingly yields three new PDEs

$$0 = a_1[rAA_{rr} - rA_r^2 + AA_r] + a_2[tAA_{rt} - tA_rA_t], \tag{C.5}$$

$$\begin{aligned} 0 = &a_1[\rho_t(rA_r^2 - rAA_{rr}) + \rho_r(rA_rA_t - rAA_{rt} - AA_t)]\\ &+ a_2[\rho_t(tA_rA_t - AA_r - tAA_{rt}) + \rho_r(tA_{t^2} - tAA_{tt})]\\ &+ a_4[\rho_rAA_t - \rho_tAA_r], \end{aligned} \tag{C.6}$$

$$0 = a_1[u_rA^2 - tAA_{rt} + rA_rA_t] - a_2[u_rA^2 + tAA_{tt} + AA + t - tA_t^2] - a_4u_rA^2. \tag{C.7}$$

Equation (C.5) corresponds to the first constraint on the cross-sectional area given by Equation (3.30). Equation (C.7) can be further treated as a polynomial in u_r leading to

$$a_1 - a_2 = a_4, \tag{C.8}$$

$$a_1[rAA_{rt} - rA_rA_t] + a_2[tAA_{tt} - tA_{t^2} + AA_t] = 0. \tag{C.9}$$

Equation (C.8) is the first constraint derived between the group parameters and Equation (C.9) corresponds to the second constraint on the area function given by Equation (3.31). Finally, Equation (C.6) can be treated as a polynomial in ρ_r leading to the following two equations

$$a_1[rA_r^2 - aAA_{rr}] + a_2[tA_rA_t - AA_r - tAA_{rt}] = a_4AA_r = 0, \tag{C.10}$$

$$a_1[rA_rA_t - rAA_{rt} - AA_t] + a_2[tA_t^2 - tAA_{tt}] + a_4AA_t = 0. \tag{C.11}$$

Substituting the result of Equation (C.8) then recovers the two constraints on the cross-sectional area that have already been determined. The determining equation is therefore solved for the continuity equation. Corresponding equations for the momentum and the energy equations are solved analogously leading to the results of Equations (3.29) and (3.32).

Appendix D. Symmetry reduction

D.1. **Mass.** Beginning with the continuity equation of System (2.1) and expanding out the partial derivatives of the area function gives

$$A\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial r}\right] + A_t\rho + A_r\rho u = 0. \tag{D.1}$$

Substituting Equations (3.43) and (3.44) for ρ and u gives

$$r^{\delta_2}\left\{A\left[\frac{\partial \hat{\rho}}{\partial t} + r^{\delta_1}\left\{\frac{1}{r}(\delta_1+\delta_2)\hat{\rho}\hat{u} + \frac{\partial(\hat{\rho}\hat{u})}{\partial r}\right\}\right] + A_t\hat{\rho} + A_r r^{\delta_1}\hat{\rho}\hat{u}\right\} = 0. \tag{D.2}$$

Finally, dividing out the r^{δ_2} factor, substituting for the partial derivatives of the functions $\hat{f}(\xi)$ using Equations (3.46) and (3.47), and multiplying through by t gives the expression presented in Equation (3.48).

D.2. **Momentum.** Similarly, expanding the momentum equation from System (2.1) yields

$$A\left[\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial r} + \frac{\partial p}{\partial r}\right] + A_t\rho u + A_r\rho u^2. \tag{D.3}$$

Substituting Equations (3.43)–(3.45) for ρ, u and p yields

$$r^{\delta_1+\delta_2}\left\{A\left[\frac{\partial(\hat{\rho}\hat{u})}{\partial t} + r^{\delta_1}\left\{(2\delta_1+\delta_2)\frac{1}{r}\left(\hat{\rho}\hat{u}^2 + \hat{p}\right) + \left(\frac{\partial(\hat{\rho}\hat{u}^2)}{\partial r} + \frac{\partial \hat{p}}{\partial r}\right)\right\}\right]\right.$$
$$\left. + \hat{u}\left(A_t\hat{\rho} + A_r r^{\delta_1}\hat{\rho}\hat{u}\right)\right\} = 0. \tag{D.4}$$

Next, dropping the factor of $r^{\delta_1+\delta_2}$, substituting for the partial derivatives of functions of the form $\hat{f}(\xi)$ and multiplying though by a factor of t gives

$$A\left[-\xi\frac{\partial(\hat{\rho}\hat{u})}{\partial \xi} + r^{\delta_1}\frac{t}{r}\left\{(2\delta_1+\delta_2)\left(\hat{\rho}\hat{u}^2+\hat{p}\right) + (1-\delta_1)\xi\left(\frac{\partial(\hat{\rho}\hat{u}^2)}{\partial \xi} + \frac{\partial \hat{p}}{\partial \xi}\right)\right\}\right]$$
$$+ \hat{u}\left(A_t t\hat{\rho} + A_r t r^{\delta_1}\hat{\rho}\hat{u}\right) = 0. \tag{D.5}$$

This equation can be re-expressed in the form

$$A\left[-\xi\hat{\rho}\frac{\partial\hat{u}}{\partial\xi}+\frac{t}{r}r^{\delta_1}\left\{\delta_1\hat{\rho}\hat{u}^2+(2\delta_1+\delta_2)\hat{p}+(1-\delta_1)\xi\frac{\partial\hat{p}}{\partial\xi}+(1-\delta_1)\xi\hat{\rho}\hat{u}\frac{\partial\hat{u}}{\partial\xi}\right\}\right]+\underbrace{\hat{u}\left\{\text{mass}\right\}}_{=0}=0, \quad \text{(D.6)}$$

where {mass} is given by Equation (3.48). Finally, dropping the factor A gives Equation (3.49).

D.3. **Energy.** In the final reduction, the energy equation of System (2.1) is expanded to get

$$\begin{aligned}A\Bigg[e\frac{\partial\rho}{\partial t}+\rho\left\{e_p\frac{\partial p}{\partial t}+e_\rho\frac{\partial\rho}{\partial t}\right\}+e\frac{\partial(\rho u)}{\partial r}+\rho u\left\{e_p\frac{\partial p}{\partial r}+e_\rho\frac{\partial\rho}{\partial r}\right\}+\frac{\partial(pu)}{\partial r}\\+\frac{1}{2}\left\{\frac{\partial(\rho u^2)}{\partial t}+\frac{\partial(\rho u^3)}{\partial r}\right\}\Bigg]+A_t\left[\rho e+\frac{\rho u^2}{2}+p\right]+A_r\left[\rho ue+pu+\frac{\rho u^3}{2}\right]=0. \quad \text{(D.7)}\end{aligned}$$

Again, substituting Equations (3.43)–(3.45) for ρ, u and p

$$\begin{aligned}r^{\delta_2}\Bigg\{A\Bigg[e\frac{\partial\hat{\rho}}{\partial t}+r^{\delta_2}\hat{\rho}\left\{e_pr^{2\delta_1}\frac{\partial\hat{p}}{\partial t}+e_\rho\frac{\partial\hat{\rho}}{\partial t}\right\}+r^{\delta_1}\Bigg\{e(\delta_1+\delta_2)\frac{1}{r}\hat{\rho}\hat{u}+e\frac{\partial(\hat{\rho}\hat{u})}{\partial r}\\+\hat{\rho}\hat{u}\left(e_pr^{2\delta_1+\delta_2}\left(\frac{1}{r}(2\delta_1+\delta_2)\hat{p}+\frac{\partial\hat{p}}{\partial r}\right)+e_\rho r^{\delta_2}\left(\frac{\delta_2}{r}\hat{\rho}+\frac{\partial\hat{\rho}}{\partial r}\right)\right)\\+r^{2\delta_1}\left(\frac{1}{r}(3\delta_1+\delta_2)\hat{p}\hat{u}+\frac{\partial(\hat{p}\hat{u})}{\partial r}\right)\\+\frac{1}{2}\left(r^{\delta_1}\frac{\partial(\hat{\rho}\hat{u}^2)}{\partial t}+r^{2\delta_1}\left(\frac{1}{r}(3\delta_1+\delta_2)\hat{\rho}\hat{u}^3+\frac{\partial(\hat{\rho}\hat{u}^3)}{\partial r}\right)\right)\Bigg\}\Bigg]\\+A_t\left[\hat{\rho}e+r^{2\delta_1}\left(\frac{\hat{\rho}\hat{u}^2}{2}+\hat{p}\right)\right]+A_r\left[r^{\delta_1}\hat{\rho}\hat{u}e+r^{3\delta_1}\left\{\hat{p}\hat{u}+\frac{\hat{\rho}\hat{u}^3}{2}\right\}\right]\Bigg\}=0. \quad \text{(D.8)}\end{aligned}$$

Dropping the r^{δ_2} factor, substituting Equations (3.46) and (3.47) for the partial derivatives and multiplying through by t

$$\begin{aligned}A\Bigg[-e\xi\frac{\partial\hat{\rho}}{\partial\xi}-r^{\delta_2}\hat{\rho}\left\{e_pr^{2\delta_1}\xi\frac{\partial\hat{p}}{\partial\xi}+e_\rho\xi\frac{\partial\hat{\rho}}{\partial\xi}\right\}+r^{\delta_1}\frac{t}{r}\Bigg\{e(\delta_1+\delta_2)\hat{\rho}\hat{u}+e(1-\delta_1)\xi\frac{\partial(\hat{\rho}\hat{u})}{\partial\xi}\\+\hat{\rho}\hat{u}\left(e_pr^{2\delta_1+\delta_2}\left((2\delta_1+\delta_2)\hat{p}+(1-\delta_1)\xi\frac{\partial\hat{p}}{\partial\xi}\right)+e_\rho r^{\delta_2}\left(\delta_2\hat{\rho}+(1-\delta_1)\xi\frac{\partial\hat{\rho}}{\partial\xi}\right)\right)\\+r^{2\delta_1}\left((3\delta_1+\delta_2)\hat{p}\hat{u}+(1-\delta_1)\xi\frac{\partial(\hat{p}\hat{u})}{\partial\xi}\right)\\+\frac{r^{2\delta_1}}{2}\left((3\delta_1+\delta_2)\hat{\rho}\hat{u}^3+(1-\delta_1)\xi\frac{\partial(\hat{\rho}\hat{u}^3)}{\partial\xi}\right)\Bigg\}-\frac{r^{2\delta_1}}{2}\xi\frac{\partial(\hat{\rho}\hat{u}^2)}{\partial\xi}\Bigg]\\+A_tt\left[\hat{\rho}e+r^{2\delta_1}\left(\frac{\hat{\rho}\hat{u}^2}{2}+\hat{p}\right)\right]+A_rt\left[r^{\delta_1}\hat{\rho}\hat{u}e+r^{3\delta_1}\left\{\hat{p}\hat{u}+\frac{\hat{\rho}\hat{u}^3}{2}\right\}\right]=0. \quad \text{(D.9)}\end{aligned}$$

This equation can be re-expressed in the form

$$\underbrace{\left(r^{2\delta_1}\frac{\hat{u}^2}{2}+e\right)\left\{\text{mass}\right\}}_{=0}+\underbrace{r^{2\delta_1}A\hat{u}\left\{\text{momentum}\right\}}_{=0}+\left\{\text{energy}\right\}. \quad \text{(D.10)}$$

where {mass}, {momentum} and {energy} are given by Equations (3.48)–(3.50) respectively.

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